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Volume-forms and minimal action principles in affine manifolds

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Abstract

Through the analysis of volume-forms in differentiable manifolds, it is shown that the usual way of defining minimal action principles for field theory on curved space–times is not appropriate on non-riemannian manifolds. An alternative approach, based in a new volume-form, is proposed and confronted with the standard one. The new volume element is explicitly used in the study of Einstein–Cartan theory of gravity and its relation to string theory, in connection with some recent results on the subject.

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This note discusses the problem of volume definition in differentiable manifolds and its relation with minimal action principles. Action principles are the starting point for several models in Physics, and they usually are formulated in non-euclidean (non-minkowskian) manifolds. In an n-dimensional manifold, the action is the integral of an n-form, but we can consider it as the integral of a scalar (the dual of an n-form) if we introduce a covariant volume element. Usually we have in a given coordinate system:

$$S = \int \mathcal{L}\, d\mathrm{vol} = \int \mathcal{L}\sqrt{|g|}\, d^n x, \tag{1}$$

where $\mathcal{L}$ is a lagrangian and $d$vol is the covariant volume element. The density $\sqrt{|g|}$ is naively introduced with the argument that it makes the euclidean volume element $d^n x$ covariant.

In spite of the fact that it is well known that there is some arbitrariness in the definition of volume elements in non-riemannian manifolds [1], the usual volume element of (1) is used for them as well. This is the case, for example, when field theory is described

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in the frame of Einstein–Cartan theory of gravity [2]. It will be shown that, for affine manifolds, there is a natural compatibility condition that a volume element should obey and that the usual one does not obey it. Such a condition will be used in order to construct compatible volume elements for general affine manifolds, and they will be used for the description of field theory on Riemann–Cartan manifolds.

In this work, $\mathcal{M}$ is an n-dimensional C^∞ differentiable oriented manifold, and $\Omega^m(\mathcal{M})$ the space of differential m-forms on it. We call $\mathcal{M}$ an affine manifold if it is endowed with a linear connection $\Gamma^\alpha_{\beta\gamma}$, which is used to define the covariant derivative of tensor valued differential forms

$$D\Pi^\alpha_\beta = d\Pi^\alpha_\beta + \omega^\alpha_\rho \wedge \Pi^\rho_\beta - \omega^\rho_\beta \wedge \Pi^\alpha_\rho - w\omega \wedge \Pi^\alpha_\beta, \tag{2}$$

where $\omega^\alpha_\beta = \Gamma^\alpha_{\mu\beta}dx^\mu$, $\omega = \Gamma^\alpha_{\alpha\mu}dx^\mu$, and w is the weight of Π^α_β. We will assume also that a metric tensor $g_{\alpha\beta}(x)$ is defined on $\mathcal{M}$ so that

$$ds^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta. \tag{3}$$

The anti-symmetrical part of the affine connection $S_{\alpha\beta}{}^\gamma = \frac{1}{2}(\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha})$, defines a new tensor, the torsion tensor. In an affine manifold, the linear connection can be written as

$$\Gamma^\alpha_{\beta\gamma} = \left\{ {\alpha \atop \beta\gamma} \right\} - K_{\beta\gamma}{}^\alpha + V^\alpha_{\beta\gamma}, \tag{4}$$

where $\left\{ {\alpha \atop \beta\gamma} \right\}$ are the Christoffel symbols, $K_{\alpha\beta}{}^\gamma$ is the contorsion tensor,

$$K_{\alpha\beta}{}^\gamma = -S_{\alpha\beta}{}^\gamma + S_\beta{}^\gamma{}_\alpha - S^\gamma{}_{\alpha\beta}, \tag{5}$$

and $V_{\alpha\beta\gamma}$ is given by:

$$V_{\alpha\beta\gamma} = \frac{1}{2}\left(D_\alpha g_{\beta\gamma} - D_\gamma g_{\alpha\beta} - D_\beta g_{\gamma\alpha}\right). \tag{6}$$

For simplicity, the traces of $S_{\nu\mu}{}^\rho$ and $V_{\nu\mu\rho}$ will be denoted by S_μ and V_μ respectively, $S_\mu = S_{\rho\mu}{}^\rho$, $V_\mu = V^\rho{}_{\rho\mu}$. An affine manifold is called a Riemann–Cartan manifold if $D_\alpha g_{\beta\gamma} = 0$, and a riemannian one if $D_\alpha g_{\beta\gamma} = 0$ and $S_{\alpha\beta\gamma} = 0$. In all these cases, the connection is said to be metric-compatible.

The main point of our discussion, the volume element, is better described by means of the concept of volume-form. A volume-form on $\mathcal{M}$ is a nowhere vanishing n-form $\upsilon \in \Omega^n(\mathcal{M})$ [4]. A volume-form, in general, can be constructed by using n linearly independent 1-forms ($\theta^1 \wedge \cdots \wedge \theta^n \neq 0$) and non-vanishing 0-forms,

$$\upsilon = f\theta^1 \wedge \cdots \wedge \theta^n. \tag{7}$$

We will assume that f is a positive non-vanishing C^∞ scalar function. The volume-form (7) defines a volume element on $\mathcal{M}$. If $\{\theta^i\}$ is assumed to be an orthonormal set of 1-forms, one has the following expression for the volume-form in general coordinates $\{x^i\}$:

$$\upsilon = f(x)\sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n. \tag{8}$$

In an affine manifold, one can require certain compatibility conditions between the affine connection and the volume-form. For a differentiable manifold with volume-form v, one usually defines the divergence of a vector field A, div A, by [3]

$$(\text{div } A)v = \pounds_A v, \tag{9}$$

where $\pounds_A$ is the Lie derivative along the direction A. However, if the manifold is endowed with an affine connection, we can define the divergence of a vector field in a very natural way by using the covariant derivative,

$$\text{div}_\Gamma A = D_\mu A^\mu. \tag{10}$$

One can use (9) and (10) to define a criterion of compatibility between the affine connection and the volume-form.

Definition 1. A volume-form v is compatible with the affine connection if

$$\pounds_A v = (D_\mu A^\mu) v, \tag{11}$$

for any vector field A on $\mathcal{M}$.

One can check that the riemannian volume-form $\mu = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n$ and the Christoffel symbols are compatible. It can be inferred also that the volume-form μ is not compatible with the connection for a Riemann–Cartan manifold,

$$\pounds_A \mu = \left(D_\mu A^\mu - 2S_\mu A^\mu \right) \mu. \tag{12}$$

The incompatibility of the usual volume-form and the connection for non-riemannian manifolds introduces a new question: Is it possible to define a volume-form compatible with the connection for non-riemannian manifolds? The answer is that sometimes it is, as we will see.

Theorem 2. *An affine manifold admits a volume-form compatible with the connection only if the form* $(V_\beta + 2S_\beta) dx^\beta$ *is exact.*

Proof. In an affine manifold with volume-form v one has

$$\pounds_A v = \left[A^\mu D_\mu \left(f(x) \sqrt{|g|} \right) + f(x) \sqrt{|g|} D_\mu A^\mu \right] dx^1 \wedge \cdots \wedge dx^n, \tag{13}$$

and in order to get (11) for arbitrary A^μ one needs

$$\begin{aligned} D_\mu (f(x) \sqrt{|g|}) &= \sqrt{|g|} \partial_\mu f(x) + f(x) \partial_\mu \sqrt{|g|} - \left\{ {}^{\rho}_{\rho\mu} \right\} f(x) \sqrt{|g|} \\ &\quad - (V_\mu + 2S_\mu) f(x) \sqrt{|g|} \\ &= 0, \end{aligned} \tag{14}$$

which leads to $\partial_\mu \ln f(x) = V_\mu + 2S_\mu$. □

The 1-form in question is closed as consequence of Poincaré's lemma. If the form $(V_\beta + 2S_\beta)dx^\beta$ is not closed, the affine manifold does not admit a compatible volume-form. The connection compatible volume-form in an affine manifold will be given by

$$v = e^{2\Theta}\sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n, \tag{15}$$

where $\partial_\mu\Theta = S_\mu + \frac{1}{2}V_\mu$, and we also have that

$$\Gamma^\rho_{\rho\mu} = \partial_\mu \ln\left(e^{2\Theta}\sqrt{|g|}\right). \tag{16}$$

For a manifold endowed with a compatible volume-form, one can define an appropriate divergence theorem.

Theorem 3. *If an affine manifold $\mathcal{M}$ is endowed with a volume-form compatible with the connection, one has the following generalized Gauss formula:*

$$\int_{\mathcal{M}} D_\mu A^\mu \, d\text{vol} = \int_{\partial\mathcal{M}} A^\mu \, d\Sigma_\mu,$$

where $d\Sigma_\mu$ is the compatible surface element, given by:

$$d\Sigma_\mu = \frac{e^{2\Theta}\sqrt{|g|}}{(n-1)!}\varepsilon_{\mu\alpha_2\alpha_3\cdots\alpha_n} dx^{\alpha_2} \wedge dx^{\alpha_3} \cdots \wedge dx^{\alpha_n}.$$

This can be easily checked by choosing

$$\mu = \frac{e^{2\Theta}\sqrt{|g|}}{(n-1)!}\varepsilon_{\alpha_1\alpha_2\cdots\alpha_n} A^{\alpha_1} dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_n},$$

and using Stokes' theorem

$$\int_{\mathcal{M}} d\mu = \int_{\partial\mathcal{M}} \mu.$$

For riemannian manifolds, the connection compatible volume-form can be obtained by using the Hodge (*) operator. The (*) is a linear operator [4]

$$* : \Omega^m(\mathcal{M}) \to \Omega^{n-m}(\mathcal{M}), \tag{17}$$

which for a Riemannian manifold has the following action on a basis vector of $\Omega^m(\mathcal{M})$:

$$\begin{aligned} &*(dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_m}) \\ &= \frac{\sqrt{|g|}}{(n-m)!}\varepsilon^{\alpha_1\cdots\alpha_m}{}_{\beta_{m+1}\cdots\beta_n} dx^{\beta_{m+1}} \wedge \cdots \wedge dx^{\beta_n}, \end{aligned} \tag{18}$$

where $\varepsilon_{\alpha_1\cdots\alpha_n}$ is the totally anti-symmetrical symbol, and $\varepsilon^{\alpha_1\cdots\alpha_m}{}_{\beta_{m+1}\cdots\beta_n}$ is constructed by using the inverse of the metric tensor. The action of (18) on the basis vector of $\Omega^0(\mathcal{M})$ gives

$$*1 = \frac{\sqrt{|g|}}{n!}\varepsilon_{\alpha_1\cdots\alpha_n} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_n}, \tag{19}$$

which is the compatible volume-form for a Riemannian manifold. We can check that if an affine manifold $\mathcal{M}$ admits a connection compatible volume-form, it can be obtained using the modified Hodge (*) operator given by

$$\begin{aligned} & {}^*(dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_m}) \\ & \quad = \frac{e^{2\Theta}\sqrt{|g|}}{(n-m)!} \varepsilon^{\alpha_1 \cdots \alpha_m}{}_{\beta_{m+1} \cdots \beta_n} dx^{\beta_{m+1}} \wedge \cdots \wedge dx^{\beta_n}. \end{aligned} \tag{20}$$

As the first application of these results, let us consider classical fields on Riemann–Cartan manifolds. If we use the usual volume element in this case, one has a paradox [5], the equations gotten by the minimal coupling [2] of the minkowskian ones and the Euler–Lagrange equations of the action gotten by the minimal coupling of the minkowskian one do not coincide. The two sets of equations will be equivalent if one uses the compatible volume element in the action formulation, and we will check it for Maxwell fields on Riemann–Cartan space–times endowed with a compatible volume element.

In order to study Maxwell's equations in a metric differentiable manifold, we introduce the electromagnetic potential 1-form

$$A = A_\alpha dx^\alpha, \tag{21}$$

and from the potential 1-form we can define the Faraday 2-form

$$F = dA = \tfrac{1}{2} F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta, \tag{22}$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the usual electromagnetic tensor. The homogeneous Maxwell equations arise naturally due to definition (22) as a consequence of Poincaré's lemma

$$dF = d(dA) = \tfrac{1}{2}\partial_\gamma F_{\alpha\beta}\, dx^\gamma \wedge dx^\alpha \wedge dx^\beta = 0. \tag{23}$$

The non-homogeneous equations in Minkowski space–time are given by

$$d\,{}^*F = 4\pi\,{}^*J, \tag{24}$$

where $J = J_\alpha dx^\alpha$ is the current 1-form, and (*) is the Hodge operator in Minkowski space–time. $^*J$ and $^*F$ are given by:

$${}^*J = \frac{1}{3!}\varepsilon_{\alpha\beta\gamma\delta} J^\alpha dx^\beta \wedge dx^\gamma \wedge dx^\delta, \tag{25}$$

$${}^*F = \tfrac{1}{4}\varepsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta}\, dx^\alpha \wedge dx^\beta. \tag{26}$$

Eq. (24) can be gotten from the minimization of the following action:

$$S = -\int \left(4\pi\,{}^*J \wedge A + \tfrac{1}{2} F \wedge {}^*F\right). \tag{27}$$

The action (27) can be cast in a covariant way by using the modified Hodge (*) operator. In this case one gets the following coordinate expression for its generally covariant generalization:

$$S = \int d^4x\, e^{2\Theta}\sqrt{-g}\left(-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + 4\pi J^{\alpha}A_{\alpha}\right). \tag{28}$$

We can check that the equations obtained from the minimization of (28) are the same we would get by casting (24) directly in a covariant way. In order to do it, we need to recall that $^*F$ given by (26) is not a scalar 2-form, but it is a relative scalar 2-form with weight -1, due to the anti-symmetrical symbol. We need to replace the exterior derivative in (24) by the covariant one according to (2), $d\,^*F \to D\,^*F = d\,^*F + \omega \wedge {}^*F$.

The use of the compatible volume element has brought two main modifications to the problem of Maxwell fields on Riemann–Cartan space–times. First, it is clear from (28) that gauge fields can interact with torsion without destroying gauge invariance [6], and second, there is no difference if one starts from the action formulation or from the equations of motion.

As another application, we can study the Einstein–Cartan theory of gravity. In such a theory, space–time is assumed to be a Riemann–Cartan manifold [2]. Its dynamical equations are gotten from a Hilbert–Einstein action, and we will consider the consequences of the use of the compatible volume element in it. We have

$$\begin{aligned} S &= -\int e^{2\Theta}\sqrt{-g}d^nx\mathcal{R} \\ &= -\int e^{2\Theta}\sqrt{-g}d^nx\left(R + 4\partial_\mu\Theta\partial^\mu\Theta - K_{\nu\rho\alpha}K^{\alpha\nu\rho}\right) + \text{surf. terms}, \end{aligned} \tag{29}$$

where Theorem 3 was used. In (29), $\mathcal{R}$ is the scalar of curvature of the Riemann–Cartan manifold, calculated by the contraction of the curvature tensor obtained using the full connection, and R is the usual riemannian scalar of curvature, obtained from the Christoffel symbols.

The similarity between (29) and the action for the dilaton gravity [7,8] is surprising. The "torsion potential" Θ can be identified with the dilaton field, and (29) can provide a geometrical interpretation for the dilaton gravity [9]. Another feature of the proposed action is that, due to the peculiar Θ-dependence of the action (29), the trace of the torsion tensor can propagate, i.e., there can exist non-vanishing solutions for torsion in the vacuum. There are several possibilities of using the new volume element, and they are now under investigation.

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